

Observational signatures of the theories beyond Horndeski

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In the approach of the effective field theory of modified gravity, we derive the equations of motion for linear perturbations in the presence of a barotropic perfect fluid on the flat isotropic cosmological background. In a simple version of Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories, which is the minimum extension of Horndeski theories, we show that a slight deviation of the tensor propagation speed squared c_t^2 from 1 generally leads to the large modification to the propagation speed squared c_s^2 of a scalar degree of freedom ϕ . This problem persists whenever the kinetic energy ρ_X of the field ϕ is much smaller than the background energy density ρ_m , which is the case for most of dark energy models in the asymptotic past. Since the scaling solution characterized by the constant ratio ρ_X/ρ_m is one way out for avoiding such a problem, we study the evolution of perturbations for a scaling dark energy model in the framework of GLPV theories in the Jordan frame. Provided the oscillating mode of scalar perturbations is fine-tuned so that it is initially suppressed, the anisotropic parameter $\eta = -\Phi/\Psi$ between the two gravitational potentials Ψ and Φ significantly deviates from 1 for c_t^2 away from 1. For other general initial conditions, the deviation of c_t^2 from 1 gives rise to the large oscillation of Ψ with the frequency related to c_s^2 . In both cases, the model can leave distinct imprints for the observations of CMB and weak lensing.

I. INTRODUCTION

The constantly accumulating observational evidence for the late-time acceleration of the Universe [1–4] implies that there may be at least one additional degree of freedom to the system of the Einstein-Hilbert action with non-relativistic matter and radiation. One simple example is a canonical scalar field ϕ with a sufficiently flat potential $V(\phi)$ —dubbed quintessence [5]. The Λ -Cold-Dark-Matter (Λ CDM) model corresponds to the non-propagating limit of quintessence (i.e., the vanishing kinetic energy) with the constant potential Λ .

The scalar degree of freedom also arises in modified gravitational theories as a result of the breaking of gauge symmetries present in General Relativity (GR) [6]. In $f(R)$ gravity, for example, the presence of non-linear terms in the 4-dimensional Ricci scalar R gives rise to the propagation of an extra gravitational scalar degree of freedom dubbed scalarons [7–9]. Provided that the functional form of $f(R)$ is well designed [10], it is possible to realize the late-time cosmic acceleration, while suppressing the propagation of the fifth force in regions of the high density through the chameleon mechanism [11].

Another well-known example of single-scalar modified gravitational theories is the covariant Galileon [12], in which the field derivatives have couplings with the Ricci scalar R and the Einstein tensor $G_{\mu\nu}$ (see Ref. [13] for the original Minkowski Galileon). In this case the field kinetic terms drive the cosmic acceleration [14, 15], while recovering the General Relativistic behavior in local regions through the Vainshtein mechanism [16].

Many dark energy models proposed in the literature (including quintessence, $f(R)$ gravity, and covariant Galileons) can be accommodated in Horndeski theories [17]—most general second-order scalar-tensor theories with a single scalar field ϕ (see also Refs. [18]). Using the linear perturbation equations of motion on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background [19], the dark energy models in the framework of Horndeski theories can be confronted with the observations of large-scale structure, CMB, and weak lensing [20]. In such general theories the screening mechanisms of the fifth force in local regions were also studied in Refs. [21].

There is another approach to the unified description of modified gravity—the effective field theory (EFT) of cosmological perturbations [22]–[36]. By means of the Arnowitt-Deser-Misner (ADM) formalism with the 3+1 decomposition of space-time [37], one can construct several geometric scalar quantities from the extrinsic curvature $K_{\mu\nu}$ and the 3-dimensional intrinsic curvature $\mathcal{R}_{\mu\nu}$, e.g., $K \equiv K^\mu{}_\mu$, $\mathcal{S} \equiv K_{\mu\nu}K^{\mu\nu}$, $\mathcal{R} \equiv \mathcal{R}^\mu{}_\mu$. The EFT of modified gravity is based on the expansion of a general Lagrangian L in unitary gauge that depends on these geometric scalars, the lapse N , and the time t . In fact, Horndeski theories can be encompassed in such a general framework with several conditions imposed for the elimination of spatial derivatives higher than second order [29].

Recently, Gleyzes *et al.* [35] proposed a generalized version of Horndeski theories by extending the Horndeski Lagrangian in the ADM form such that two additional constraints are not imposed. For example the Horndeski Lagrangian involves the term $L_4 = A_4(K^2 - \mathcal{S}) + B_4\mathcal{R}$, where the functions A_4 and B_4 , which depend on ϕ and

$X = \partial_\mu \phi \partial^\mu \phi$, have a particular relation $A_4 = 2X(\partial B_4/\partial X) - B_4$. In GR the ADM decomposition of the Einstein-Hilbert term $(M_{\text{pl}}^2/2)R$, where M_{pl} is the reduced Planck mass, leads to the Lagrangian L_4 with $B_4 = -A_4 = M_{\text{pl}}^2/2$. The theories with $B_4 \neq -A_4$ belong to a class of GLPV theories.

On the flat FLRW background, the Hamiltonian analysis based on linear cosmological perturbations shows that GLPV theories have only one scalar propagating degree of freedom [35, 38–40]. One distinguished feature of GLPV theories is that the scalar and matter sound speeds are coupled to each other [34, 35]. For example, in the covariantized version of the Minkowski Galileon where partial derivatives in the Lagrangian are replaced by covariant derivatives, the scalar propagation speed squared c_s^2 becomes negative in the matter-dominated epoch due to a non-trivial kinetic-type coupling [41]. Unlike the covariant Galileon with positive c_s^2 during the matter era, the covariantized Galileon mentioned above (a class of GLPV theories) is practically excluded as a viable dark energy scenario.

In this paper, we develop the analysis of cosmological perturbations further to confront dark energy models in GLPV theories with observations. First, the linear perturbation equations of motion are derived in the presence of a barotropic perfect fluid for a general Lagrangian encompassing GLPV theories. We provide a convenient analytic formula for c_s^2 and show that even a slight deviation from Horndeski theories generally gives rise to a non-negligible modification to the scalar sound speed.

We also apply our general formalism to a simple dark energy model with a canonical scalar field ϕ in which the function B_4 differs from $-A_4 = M_{\text{pl}}^2/2$. In this case the tensor propagation speed squared $c_t^2 = -B_4/A_4$ is different from 1. We show that this deviation leads to a significant modification to c_s^2 whenever the field kinetic energy ρ_X is suppressed relative to the background energy density ρ_m . The scaling solution characterized by the constant ρ_X/ρ_m is a possible way out to avoid having large values of c_s^2 in the early cosmological epoch.

For the scaling dark energy model described by the potential $V(\phi) = V_1 e^{-\lambda_1 \phi/M_{\text{pl}}} + V_2 e^{-\lambda_2 \phi/M_{\text{pl}}}$ ($\lambda_1 \gtrsim 10$ and $\lambda_2 \lesssim 1$), we study the evolution of cosmological perturbations and resulting observational consequences. For the initial conditions where the contribution of the oscillating mode $V_m^{(h)}$ to the velocity potential V_m is suppressed, the evolution of perturbations is analytically known during the scaling matter era. In particular, the anisotropic parameter $\eta = -\Phi/\Psi$ between the two gravitational potentials Ψ and Φ exhibits a large deviation from 1 for c_t^2 away from 1. If the oscillating mode $V_m^{(h)}$ gives a non-negligible contribution to V_m initially, the rapid oscillations with frequencies related to c_s^2 arise for the perturbations like Ψ and V_m . Thus the model in the framework of GLPV theories can be clearly distinguished from that in Horndeski theories.

This paper is organized as follows. In Sec. II the extension of Horndeski theories to GLPV theories is briefly reviewed. In Sec. III the perturbation equations of motion are derived in the presence of a barotropic perfect fluid according to the EFT approach encompassing GLPV theories. In Sec. IV we obtain convenient formulae for the matter and scalar propagation speeds in the small-scale limit. In Sec. V we present observables associated with the measurements of large-scale structure, CMB, weak lensing, and discuss the quasi-static approximation on sub-horizon scales. In Sec. VI we propose a simple dark energy model in the framework of GLPV theories and study its observational signatures by carefully paying attention to the oscillating mode induced by c_s^2 . Sec. VII is devoted to conclusions.

II. THE THEORIES BEYOND HORNDESKI

The EFT of cosmological perturbations is a powerful framework to deal with low-energy degrees of freedom in a systematic and unified way for a wide variety of modified gravity theories. It is based upon the 3+1 ADM decomposition of space-time described by the line element [37]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (2.1)$$

where N is the lapse function, N^i is the shift vector, and h_{ij} is the three-dimensional spatial metric. The extrinsic curvature is defined by $K_{\mu\nu} = h_{\mu}^\lambda n_{\nu;\lambda}$, where $n_\mu = (-N, 0, 0, 0)$ is a unit vector orthogonal to the constant t hypersurfaces Σ_t and a semicolon represents a covariant derivative. We also introduce the three-dimensional Ricci tensor $\mathcal{R}_{\mu\nu} = {}^{(3)}R_{\mu\nu}$ on Σ_t . Then we can construct a number of geometric scalar quantities:

$$K \equiv K^\mu{}_\mu, \quad S \equiv K_{\mu\nu} K^{\mu\nu}, \quad \mathcal{R} \equiv \mathcal{R}^\mu{}_\mu, \quad \mathcal{Z} \equiv \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}, \quad \mathcal{U} \equiv \mathcal{R}_{\mu\nu} K^{\mu\nu}. \quad (2.2)$$

Horndeski theories [17] are the most general scalar-tensor theories with second-order equations of motion in generic space-time. The action of Horndeski theories is given by $S = \int d^4x \sqrt{-g} L$ with the Lagrangian [18]

$$\begin{aligned} L = & G_2(\phi, X) + G_3(\phi, X) \square \phi + G_4(\phi, X) R - 2G_{4,X}(\phi, X) [(\square \phi)^2 - \phi^{;\mu\nu} \phi_{;\mu\nu}] \\ & + G_5(\phi, X) G_{\mu\nu} \phi^{;\mu\nu} + \frac{1}{3} G_{5,X}(\phi, X) [(\square \phi)^3 - 3(\square \phi) \phi_{;\mu\nu} \phi^{;\mu\nu} + 2\phi_{;\mu\nu} \phi^{;\mu\sigma} \phi^{;\nu}{}_{;\sigma}], \end{aligned} \quad (2.3)$$

where $\square\phi \equiv (g^{\mu\nu}\phi_{;\nu})_{;\mu}$, and the four functions G_i ($i = 2, 3, 4, 5$) depend on a scalar field ϕ and its kinetic energy $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, with $G_{i,X} \equiv \partial G_i/\partial X$. Choosing the unitary gauge $\phi = \phi(t)$ on the flat FLRW background described by the line element $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, the Lagrangian (2.3) can be expressed in terms of the geometric scalars introduced above, as [29]

$$L = A_2(N, t) + A_3(N, t)K + A_4(N, t)(K^2 - \mathcal{S}) + B_4(N, t)\mathcal{R} + A_5(N, t)K_3 + B_5(N, t)(\mathcal{U} - K\mathcal{R}/2), \quad (2.4)$$

where $K_3 \equiv K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\lambda}K^\nu{}_\lambda$. Up to quadratic order in the perturbations we have $K_3 = 3H(2H^2 - 2KH + K^2 - \mathcal{S})$, where $H = \dot{a}/a$ is the Hubble parameter (a dot represents a derivative with respect to t).

Horndeski theories satisfy the following two conditions [29]

$$A_4 = 2XB_{4,X} - B_4, \quad A_5 = -\frac{1}{3}XB_{5,X}. \quad (2.5)$$

More concretely, the coefficients G_i in Eq. (2.3) and A_i, B_i in Eq. (2.4) are related with each other, as

$$\begin{aligned} A_2 &= G_2 - XF_{3,\phi}, & A_3 &= 2(-X)^{3/2}F_{3,X} - 2\sqrt{-X}G_{4,\phi}, \\ A_4 &= -G_4 + 2XG_{4,X} + XG_{5,\phi}/2, & B_4 &= G_4 + X(G_{5,\phi} - F_{5,\phi})/2, \\ A_5 &= -(-X)^{3/2}G_{5,X}/3, & B_5 &= -\sqrt{-X}F_5, \end{aligned} \quad (2.6)$$

where F_3 and F_5 are auxiliary functions obeying the relations $G_3 = F_3 + 2XF_{3,X}$ and $G_{5,X} = F_5/(2X) + F_{5,X}$. Since $X = -\dot{\phi}^2(t)/N^2$ in unitary gauge, the functional dependence of ϕ and X can translate to that of t and N .

It is possible to go beyond the Horndeski domain without imposing the two conditions (2.5) [35]. This generally gives rise to derivatives higher than second order, but it does not necessarily mean that an extra propagating degree of freedom is present. In fact, the Hamiltonian analysis on the flat FLRW background shows that the theories described by the Lagrangian (2.4), dubbed GLPV theories, do not possess an extra scalar mode of the propagation [35, 38–40].

III. PERTURBATION EQUATIONS OF MOTION

The Lagrangian (2.4) depends on $N, K, \mathcal{S}, \mathcal{R}, \mathcal{U}, t$, but not on \mathcal{Z} . The dependence on \mathcal{Z} appears in the theories with spatial derivatives higher than second order [33, 36], e.g., in Hořava-Lifshitz gravity [42]. In the following we shall focus on the theories described by the action

$$S = \int d^4x \sqrt{-g} L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{U}; t) + \int d^4x \sqrt{-g} L_m(g_{\mu\nu}, \Psi_m), \quad (3.1)$$

where L_m is the Lagrangian of the matter field Ψ_m . We consider a metric frame in which the scalar field ϕ is not directly coupled to matter (dubbed the Jordan frame). For the matter component we consider a scalar field χ characterized by

$$L_m = P(Y), \quad Y = g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi, \quad (3.2)$$

whose description is the same as that of the barotropic perfect fluid [43–45]. The perfect fluids of radiation and non-relativistic matter can be modeled by $P(Y) = c_1 Y^2$ and $P(Y) = c_2(Y - Y_0)^2$ with $|(Y - Y_0)/Y_0| \ll 1$, respectively, where c_1, c_2, Y_0 are constants [41, 46].

The linearly perturbed line element on the flat FLRW background with four metric perturbations A, ψ, ζ, E and tensor perturbations γ_{ij} is given by [47]

$$ds^2 = -(1 + 2A)dt^2 + 2\partial_i\psi dt dx^i + a^2(t)[(1 + 2\zeta)\delta_{ij} + 2\partial_i\partial_j E + \gamma_{ij}]dx^i dx^j. \quad (3.3)$$

Then the shift vector N_i is related to the perturbation ψ , as

$$N_i = \partial_i\psi. \quad (3.4)$$

In the following we choose the gauge conditions

$$\delta\phi = 0, \quad E = 0, \quad (3.5)$$

under which temporal and spatial components of the gauge-transformation vector ξ^μ are fixed. The background values (denoted by an overbar) of the ADM geometric quantities are

$$\bar{K}_{\mu\nu} = H\bar{h}_{\mu\nu}, \quad \bar{K} = 3H, \quad \bar{\mathcal{S}} = 3H^2, \quad \bar{\mathcal{R}}_{\mu\nu} = 0, \quad \bar{\mathcal{R}} = \bar{\mathcal{U}} = 0. \quad (3.6)$$

Around the FLRW background we consider the scalar perturbations $\delta N = N - 1$, $\delta K_{\mu\nu} = K_{\mu\nu} - H h_{\mu\nu}$, $\delta S = 2H\delta K + \delta K^\mu{}_\nu \delta K^\nu{}_\mu$, and $\mathcal{R} = \delta_1 \mathcal{R} + \delta_2 \mathcal{R}$, where $\delta_1 \mathcal{R}$ and $\delta_2 \mathcal{R}$ are the first-order and second-order perturbations respectively. The scalar \mathcal{U} , which is a perturbed quantity itself, obeys the relation $\int d^4x \sqrt{-g} \alpha(t) \mathcal{U} = \int d^4x \sqrt{-g} [\alpha(t) \mathcal{R} K / 2 + \dot{\alpha}(t) \mathcal{R} / (2N)]$ up to a boundary term, where $\alpha(t)$ is an arbitrary function with respect to t .

Decomposing the scalar field χ as $\chi = \bar{\chi}(t) + \delta\chi(t, \mathbf{x})$ and omitting the overbar in the following discussion, the kinetic term Y , expanded up to second order, can be expressed in the form $Y = -\dot{\chi}^2 + \delta_1 Y + \delta_2 Y$, where

$$\delta_1 Y = 2\dot{\chi}^2 \delta N - 2\dot{\chi} \delta\dot{\chi}, \quad (3.7)$$

$$\delta_2 Y = -\dot{\chi}^2 - 3\dot{\chi}^2 \delta N^2 + 4\dot{\chi} \delta\dot{\chi} \delta N + \frac{2\dot{\chi}}{a^2} \delta^{ij} \partial_i \psi \partial_j \delta\chi + \frac{1}{a^2} (\partial \delta\chi)^2, \quad (3.8)$$

and $(\partial \delta\chi)^2 \equiv \delta^{ij} \partial_i \delta\chi \partial_j \delta\chi$. The energy-momentum tensor of the field χ is given by $T_{\mu\nu} = P g_{\mu\nu} - 2P_{,Y} \partial_\mu \chi \partial_\nu \chi$. Defining the linear perturbations of energy density, momentum, and pressure, respectively, as $\delta T_0^0 = -\delta\rho$, $\delta T_i^0 = \partial_i \delta q$, and $\delta T_j^i = \delta P \delta_j^i$, it follows that

$$\delta\rho = (P_{,Y} + 2Y P_{,YY}) \delta_1 Y, \quad \delta q = 2P_{,Y} \dot{\chi} \delta\chi, \quad \delta P = P_{,Y} \delta_1 Y. \quad (3.9)$$

These quantities appear in the perturbation equations of motion presented later.

A. Background equations of motion

Expanding the action (3.1) up to linear order in scalar perturbations, we obtain the first-order action $S^{(1)} = \int d^4x \mathcal{L}_1$ with [29, 34]

$$\mathcal{L}_1 = a^3 (\bar{L} + L_{,N} - 3H\mathcal{F} - \rho) \delta N + 3 (\bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + P) a^2 \delta a - 2a^3 P_{,Y} \dot{\chi} \delta\chi + a^3 \mathcal{E} \delta_1 \mathcal{R}, \quad (3.10)$$

where

$$\mathcal{F} \equiv L_{,K} + 2H L_{,S}, \quad (3.11)$$

$$\mathcal{E} \equiv L_{,\mathcal{R}} + \frac{1}{2} \dot{L}_{,\mathcal{U}} + \frac{3}{2} H L_{,\mathcal{U}}, \quad (3.12)$$

$$\rho \equiv 2Y P_{,Y} - P. \quad (3.13)$$

The last term of Eq. (3.10) is a total derivative irrelevant to the background dynamics. Varying Eq. (3.10) with respect to δN , δa , and $\delta\chi$, respectively, we obtain the background equations of motion

$$\bar{L} + L_{,N} - 3H\mathcal{F} = \rho, \quad (3.14)$$

$$\bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} = -P, \quad (3.15)$$

$$\frac{d}{dt} (a^3 P_{,Y} \dot{\chi}) = 0. \quad (3.16)$$

These correspond to the Hamiltonian constraint, the momentum constraint, and the equation of motion for χ , respectively. Equation (3.16) is equivalent to the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$ by using the definition (3.13) of the field energy density.

B. Perturbation equations of motion

We expand the action (3.1) up to second order in scalar perturbations to derive the linear perturbation equations of motion. In doing so, we use the following properties

$$\delta K_j^i = \left(\dot{\zeta} - H \delta N \right) \delta_j^i - \frac{1}{2a^2} \delta^{ik} (\partial_k N_j + \partial_j N_k), \quad (3.17)$$

$$\delta \mathcal{R}_{ij} = -(\delta_{ij} \partial^2 \zeta + \partial_i \partial_j \zeta), \quad (3.18)$$

where $\partial^2 \zeta \equiv \delta^{ij} \partial_i \partial_j \zeta$. Expansion of the action (3.1) gives rise to the terms in the forms $(\partial^2 \psi)^2 / a^4$, $(\partial^2 \psi)(\partial^2 \zeta) / a^4$, and $(\partial^2 \zeta)^2 / a^4$, which generate the spatial derivatives higher than second order. The absence of these higher-order

terms requires that [29, 34]

$$L_{,KK} + 4HL_{,SK} + 4H^2L_{,SS} + 2L_{,S} = 0, \quad (3.19)$$

$$L_{,K\mathcal{R}} + 2HL_{,S\mathcal{R}} + \frac{1}{2}L_{,\mathcal{U}} + HL_{,KU} + 2H^2L_{,SU} = 0, \quad (3.20)$$

$$L_{,\mathcal{R}\mathcal{R}} + 2HL_{,\mathcal{R}\mathcal{U}} + H^2L_{,\mathcal{U}\mathcal{U}} = 0. \quad (3.21)$$

The GLPV Lagrangian (2.4) obeys all these conditions, so we assume the conditions (3.19)-(3.21) in the following discussion.

The second-order Lagrangian density of $\mathcal{L}_m = \sqrt{-g}P(Y) = N\sqrt{h}P(Y)$ reads

$$\mathcal{L}_m^{(2)} = -\rho\delta N\delta\sqrt{h} - 2P_{,Y}\dot{\chi}\delta\chi\delta\sqrt{h} + a^3\left(P_{,Y}\delta_2Y + \frac{1}{2}P_{,YY}\delta_1Y^2 + P_{,Y}\delta N\delta_1Y\right), \quad (3.22)$$

where $\delta\sqrt{h} = 3a^3\zeta$. The Lagrangian density $\mathcal{L} = \sqrt{-g}L$ contains the second-order contribution $(\bar{L} + L_{,N} - 3H\mathcal{F})\delta N\delta\sqrt{h}$. On using the background Eq. (3.14), this cancels the first term of Eq. (3.22). Then, expansion of the total action (3.1) leads to the second-order action $S^{(2)} = \int d^4x \mathcal{L}_T^{(2)}$ with

$$\begin{aligned} \mathcal{L}_T^{(2)} = a^3 & \left[\left\{ L_{,N} + \frac{1}{2}L_{,NN} - 3H(\mathcal{W} - 2L_{,SH}) \right\} \delta N^2 + \left\{ \mathcal{W} \left(3\dot{\zeta} - \frac{\partial^2\psi}{a^2} \right) - 4(\mathcal{D} + \mathcal{E}) \frac{\partial^2\zeta}{a^2} \right\} \delta N + 4L_{,S}\dot{\zeta} \frac{\partial^2\psi}{a^2} \right. \\ & \left. - 6L_{,S}\dot{\zeta}^2 + 2\mathcal{E} \frac{(\partial\zeta)^2}{a^2} - (P_{,Y} + 2YP_{,YY})(\dot{\chi}\delta N - \delta\dot{\chi})^2 - 6P_{,Y}\dot{\chi}\delta\dot{\chi}\zeta - 2P_{,Y}\dot{\chi}\delta\chi \frac{\partial^2\psi}{a^2} + P_{,Y} \frac{(\partial\delta\chi)^2}{a^2} \right], \quad (3.23) \end{aligned}$$

where

$$\mathcal{W} \equiv L_{,KN} + 2HL_{,SN} + 4L_{,SH}, \quad (3.24)$$

$$\mathcal{D} \equiv L_{,N\mathcal{R}} - \frac{1}{2}\dot{L}_{,\mathcal{U}} + HL_{,NU}. \quad (3.25)$$

Varying Eq. (3.23) with respect to δN , $\partial^2\psi$, ζ , and $\delta\chi$ respectively, and employing Eqs. (3.9) and (3.16), we obtain the following linear perturbation equations of motion

$$(2L_{,N} + L_{,NN} - 6H\mathcal{W} + 12L_{,SH}^2)\delta N + \left(3\dot{\zeta} - \frac{\partial^2\psi}{a^2} \right) \mathcal{W} - 4(\mathcal{D} + \mathcal{E}) \frac{\partial^2\zeta}{a^2} = \delta\rho, \quad (3.26)$$

$$\mathcal{W}\delta N - 4L_{,S}\dot{\zeta} = -\delta q, \quad (3.27)$$

$$\frac{1}{a^3} \frac{d}{dt} (a^3\mathcal{Y}) + 4(\mathcal{D} + \mathcal{E}) \frac{\partial^2\delta N}{a^2} + \frac{4\mathcal{E}}{a^2} \partial^2\zeta + 6P_{,Y}\dot{\chi}^2\delta N = 3\delta P, \quad (3.28)$$

$$\frac{1}{a^3} \frac{d}{dt} \left[a^3(P_{,Y} + 2YP_{,YY})(\delta\dot{\chi} - \dot{\chi}\delta N) \right] + 3P_{,Y}\dot{\chi}\dot{\zeta} - \dot{\chi}P_{,Y} \frac{\partial^2\psi}{a^2} - P_{,Y} \frac{\partial^2\delta\chi}{a^2} = 0, \quad (3.29)$$

where

$$\mathcal{Y} \equiv 4L_{,S} \frac{\partial^2\psi}{a^2} - 3\delta q. \quad (3.30)$$

On using Eq. (3.16), it is easy to show that the momentum perturbation $\delta q = 2P_{,Y}\dot{\chi}\delta\chi$ obeys

$$\dot{\delta q} + 3H\delta q = -(\rho + P)\delta N - \delta P. \quad (3.31)$$

Similarly, Eq. (3.29) can be expressed in the following form

$$\dot{\delta\rho} + 3H(\delta\rho + \delta P) = -(\rho + P) \left(3\dot{\zeta} - \frac{\partial^2\psi}{a^2} \right) - \frac{\partial^2\delta q}{a^2}. \quad (3.32)$$

We note that Eqs. (3.31) and (3.32) also follow from the continuity equations $\delta T^\mu_{i;\mu} = 0$ and $\delta T^\mu_{0;\mu} = 0$, respectively. Substituting Eq. (3.30) into Eq. (3.28) and using Eq. (3.31), it follows that

$$(\dot{L}_{,S} + HL_{,S})\psi + L_{,S}\dot{\psi} + (\mathcal{D} + \mathcal{E})\delta N + \mathcal{E}\zeta = 0, \quad (3.33)$$

where the integration constant is set to 0. The dynamics of scalar perturbations is known by solving Eqs. (3.26), (3.27), (3.33) together with Eqs. (3.31) and (3.32).

For the tensor perturbation γ_{ij} the second-order action reads $S_h^{(2)} = \int d^4x \mathcal{L}_h^{(2)}$, where [29, 32, 48]

$$\mathcal{L}_h^{(2)} = a^3 \frac{L_{,S}}{4} \delta^{ik} \delta^{jl} \left(\dot{\gamma}_{ij} \dot{\gamma}_{kl} - \frac{c_t^2}{a^2} \partial \gamma_{ij} \partial \gamma_{kl} \right). \quad (3.34)$$

Here, the propagation speed squared is

$$c_t^2 = \frac{\mathcal{E}}{L_{,S}}. \quad (3.35)$$

Then the equation of motion for gravitational waves is given by

$$\ddot{\gamma}_{ij} + \left(3H + \frac{\dot{L}_{,S}}{L_{,S}} \right) \dot{\gamma}_{ij} - c_t^2 \frac{\partial^2 \gamma_{ij}}{a^2} = 0. \quad (3.36)$$

Provided that $L_{,S} > 0$ and $c_t^2 > 0$, the ghost and Laplacian instabilities are absent for the tensor mode.

IV. PROPAGATION SPEEDS OF SCALAR PERTURBATIONS

We derive the propagation speeds of the gravitational scalar and the matter field in the small-scale limit. In doing so, we first express δN and $\partial^2 \psi / a^2$ in terms of ζ , $\delta \chi$ and their derivatives by using Eqs. (3.26) and (3.27). Substituting these relations into Eq. (3.23), the Lagrangian density can be expressed in the form [34, 41]

$$\mathcal{L}_2 = a^3 \left(\vec{\mathcal{X}}^t \mathbf{K} \vec{\mathcal{X}} - \partial_j \vec{\mathcal{X}}^t \mathbf{G} \partial^j \vec{\mathcal{X}} - \vec{\mathcal{X}}^t \mathbf{B} \vec{\mathcal{X}} - \vec{\mathcal{X}}^t \mathbf{M} \vec{\mathcal{X}} \right), \quad (4.1)$$

where \mathbf{K} , \mathbf{G} , \mathbf{B} , \mathbf{M} are 2×2 matrices and $\vec{\mathcal{X}}^t = (\zeta, \delta \chi / M_{\text{pl}})$. The components of the two matrices \mathbf{K} and \mathbf{G} are given, respectively, by

$$\begin{aligned} K_{11} &= Q_s + \frac{16L_{,S}^2}{M_{\text{pl}}^2 \mathcal{W}^2} \dot{\chi}^2 K_{22}, & K_{22} &= (2\dot{\chi}^2 P_{,YY} - P_{,Y}) M_{\text{pl}}^2, & K_{12} &= K_{21} = -\frac{4L_{,S} \dot{\chi}}{M_{\text{pl}} \mathcal{W}} K_{22}, \\ G_{11} &= 2(\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E}), & G_{22} &= -P_{,Y} M_{\text{pl}}^2, & G_{12} &= G_{21} = -\frac{\mathcal{M} \dot{\chi}}{L_{,S} M_{\text{pl}}} G_{22}, \end{aligned} \quad (4.2)$$

where

$$Q_s \equiv 6L_{,S} + \frac{8L_{,S}^2}{\mathcal{W}^2} (2L_{,N} + L_{,NN} - 6HW + 12H^2 L_{,S}), \quad (4.3)$$

$$\mathcal{M} \equiv \frac{4L_{,S}(\mathcal{D} + \mathcal{E})}{\mathcal{W}}. \quad (4.4)$$

The scalar ghosts are absent as long as the determinants of principal sub-matrices of \mathbf{K} are positive, which translates to the two conditions $K_{11} > 0$ and $Q_s K_{22} > 0$. These conditions are satisfied for $Q_s > 0$ and $K_{22} > 0$.

The dispersion relation following from Eq. (4.1) in the limit of the large wave number k is given by $\det(\omega^2 \mathbf{K} - k^2 \mathbf{G} / a^2) = 0$. The scalar propagation speed c_s , which is related to the frequency ω as $\omega^2 = c_s^2 k^2 / a^2$ obeys

$$(c_s^2 K_{11} - G_{11})(c_s^2 K_{22} - G_{22}) - (c_s^2 K_{12} - G_{12})^2 = 0. \quad (4.5)$$

In Horndeski theories there is the specific relation $\mathcal{D} + \mathcal{E} = L_{,S}$ and hence $G_{12}/K_{12} = G_{22}/K_{22}$. In this case the two solutions to c_s^2 are given by

$$c_m^2 = \frac{G_{22}}{K_{22}} = \frac{P_{,Y}}{P_{,Y} - 2\dot{\chi}^2 P_{,YY}} = \frac{\delta P}{\delta \rho}, \quad (4.6)$$

$$c_H^2 = \frac{1}{Q_s} \left[G_{11} - (K_{11} - Q_s) \frac{G_{22}}{K_{22}} \right] = \frac{2}{Q_s} \left(\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E} + \frac{8L_{,S}^2 \dot{\chi}^2 P_{,Y}}{\mathcal{W}^2} \right). \quad (4.7)$$

In GLPV theories the relation $\mathcal{D} + \mathcal{E} = L_{,S}$ no longer holds, so we define the parameter [39]

$$\alpha_H \equiv \frac{\mathcal{D} + \mathcal{E}}{L_{,S}} - 1 = c_t^2 - 1 + \frac{\mathcal{D}}{L_{,S}}, \quad (4.8)$$

which characterizes the deviation from Horndeski theories. We also introduce the following quantity

$$\beta_H \equiv 2c_m^2 \left(\frac{K_{11}}{Q_s} - 1 \right) \alpha_H = \frac{16L_{,S}^2(\rho + P)}{\mathcal{W}^2 Q_s} \alpha_H. \quad (4.9)$$

If $\mathcal{D} = 0$, then the parameter α_H is simply related to the deviation of the tensor propagation speed squared from 1, as $\alpha_H = c_t^2 - 1$. This is the case for the theories with $B_4 = B_4(t)$ and constant B_5 . In Sec. VI we shall discuss the dynamics of cosmological perturbations for a simple model satisfying the condition $\mathcal{D} = 0$.

Eliminating the terms G_{22} , G_{11} , G_{12} , and K_{11} in Eq. (4.5) with the help of Eqs. (4.6), (4.7), (4.9) and the relation $G_{12}/K_{12} = (1 + \alpha_H)G_{22}/K_{22}$, the two solutions to Eq. (4.5) can be expressed as

$$\tilde{c}_m^2 = \frac{1}{2} \left[c_m^2 + c_H^2 - \beta_H + \sqrt{(c_m^2 - c_H^2 + \beta_H)^2 + 2c_m^2 \alpha_H \beta_H} \right], \quad (4.10)$$

$$c_s^2 = \frac{1}{2} \left[c_m^2 + c_H^2 - \beta_H - \sqrt{(c_m^2 - c_H^2 + \beta_H)^2 + 2c_m^2 \alpha_H \beta_H} \right]. \quad (4.11)$$

For non-relativistic matter with $c_m^2 = 0$, Eqs. (4.10) and (4.11) reduce to $\tilde{c}_m^2 = 0$ and $c_s^2 = c_H^2 - \beta_H^2$ respectively.

For the general perfect fluid with $c_m^2 \neq 0$, we consider the case in which the deviation from Horndeski theories is small, i.e., $|\alpha_H| \ll 1$. Then the propagation speeds (4.10) and (4.11) are approximately given, respectively, by

$$\tilde{c}_m^2 \simeq c_m^2 - \frac{c_m^2}{2(c_H^2 - c_m^2 - \beta_H)} \alpha_H \beta_H, \quad (4.12)$$

$$c_s^2 \simeq c_H^2 - \beta_H + \frac{c_m^2}{2(c_H^2 - c_m^2 - \beta_H)} \alpha_H \beta_H. \quad (4.13)$$

The condition, $|\alpha_H| \ll 1$, does not necessarily mean that $|\beta_H|$ is also much smaller than 1. In the covariantized Galileon model [35] where the partial derivatives of the original Minkowski Galileon [13] are replaced by the covariant derivatives, we have $\beta_H = 3/10$ and $c_H^2 = 11/40$ during the matter era for late-time tracking solutions (in which regime $|\alpha_H|$ is much smaller than 1) [41]. The reason why $|\beta_H|$ is not as small as $|\alpha_H|$ comes from the fact that the variable Q_s/M_{pl}^2 in Eq. (4.9) is much smaller than 1 in the early cosmological epoch. Since $c_s^2 = -1/40$ in this case, the covariantized Galileon is plagued by the Laplacian instability on small scales. On the other hand, for the covariant Galileon [18], we have $c_s^2 = c_H^2 = 1/40$ during the matter era [15].

Generally, the sound speed squared c_s^2 is subject to the modification arising from the deviation from Horndeski theories, such that $c_s^2 \simeq c_H^2 - \beta_H$. Meanwhile, provided that $|\alpha_H \beta_H| \ll 1$, the correction to the matter sound speed squared c_m^2 , i.e., the second term on the r.h.s. of Eq. (4.12), is suppressed to be small. This shows that the effect beyond Horndeski theories arises for the scalar sound speed c_s rather than the matter sound speed \tilde{c}_m . In Sec. VI we shall apply the results in this section to a concrete model that belongs to a class of GLPV theories.

V. CONFRONTATIONS WITH OBSERVATIONS

In this section we discuss several physical quantities associated with the measurements of large-scale structures, CMB, and weak lensing in order to confront GLPV theories with observations. We then proceed to the discussion of the quasi-static approximation for the perturbations deep inside the sound horizon.

A. Observables

We first introduce the gauge-invariant combinations of the matter density contrast δ_m and the velocity perturbation v_m , as

$$\delta_m \equiv \delta - 3Hv, \quad v_m \equiv v + (1 + w) \frac{\delta \phi}{\phi}, \quad (5.1)$$

where

$$\delta \equiv \frac{\delta\rho}{\rho}, \quad v \equiv \frac{\delta q}{\rho}, \quad w \equiv \frac{P}{\rho}. \quad (5.2)$$

Since we choose the unitary gauge ($\delta\phi = 0$), the perturbation v_m is equivalent to v itself. In Fourier space we can rewrite Eqs. (3.31) and (3.32), respectively, as

$$\dot{v}_m + 3H(c_m^2 - w)v_m = -(1+w)\delta N - c_m^2\delta_m, \quad (5.3)$$

$$\dot{\delta}_m + 3(Hv_m)' + 3H(c_m^2 - w)(\delta_m + 3Hv_m) = -(1+w)\left(3\dot{\zeta} + \frac{k^2}{a^2}\psi\right) + \frac{k^2}{a^2}v_m, \quad (5.4)$$

where c_m is defined by Eq. (4.6).

Since we are interested in the growth of structures after the onset of the matter era, we shall focus on the case of non-relativistic matter characterized by $w = 0$ and $c_m^2 = 0$. Taking the time derivative of Eq. (5.4) and using Eq. (5.3), we obtain

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Psi = -3\ddot{B} - 6H\dot{B}, \quad (5.5)$$

where $B \equiv \zeta + Hv_m$, and Ψ is the gauge-invariant gravitational potential defined by [47]

$$\Psi \equiv \delta N + \dot{\psi}. \quad (5.6)$$

If c_m is not exactly 0 and the term $-c_m^2\delta_m$ on the r.h.s. of Eq. (5.3) is not neglected, this gives rise to the term $c_m^2(k^2/a^2)\delta_m$ on the l.h.s. of Eq. (5.5). This works as a pressure that prevents the gravitational growth induced by the source term $(k^2/a^2)\Psi$. The matter propagation speed squared \tilde{c}_m^2 in GLPV theories is not equivalent to c_m^2 , but, in the limit $c_m^2 \rightarrow 0$, they are identical to each other. On the other hand, the scalar sound speed squared c_s^2 is generally subject to a non-negligible change even by the slight deviation from Horndeski theories.

In order to know the evolution of the matter density contrast δ_m , we need to relate the gravitational potential Ψ in (5.5) with δ_m . Usually, this relation is expressed in the following form

$$\frac{k^2}{a^2}\Psi = -4\pi G_{\text{eff}}\rho\delta_m, \quad (5.7)$$

where G_{eff} is the effective gravitational coupling. In GR, G_{eff} is equivalent to the Newton gravitational constant G . In modified gravitational theories, G_{eff} generally differs from G . In Horndeski theories, for example, the quasi-static approximation provides the analytic expression of G_{eff} for the perturbations deep inside the sound horizon [19]. Provided that the terms on the r.h.s. of Eq. (5.5) are negligible compared to those on the l.h.s., the evolution of δ_m is known by integrating Eq. (5.5). The growth rate of δ_m is related to the peculiar velocity of galaxies [49]. For the observations of redshift-space distortions, the quantity $f\sigma_8$ is usually introduced [50], where

$$f \equiv \frac{\dot{\delta}_m}{H\delta_m}, \quad (5.8)$$

and σ_8 is the amplitude of over-density at the comoving $8h^{-1}$ Mpc scale (h is the normalized Hubble constant $H_0 = 100h \text{ km sec}^{-1} \text{ Mpc}^{-1}$).

In order to confront modified gravity models with the observations of CMB and weak lensing, we also introduce the following gauge-invariant gravitational potential

$$\Phi \equiv \zeta + H\psi. \quad (5.9)$$

The effective gravitational potential associated with the deviation of light rays is given by [51]

$$\Phi_{\text{eff}} \equiv \frac{1}{2}(\Psi - \Phi). \quad (5.10)$$

Introducing the anisotropic parameter

$$\eta \equiv -\frac{\Phi}{\Psi}, \quad (5.11)$$

Eq. (5.10) can be expressed as $\Phi_{\text{eff}} = (1 + \eta)\Psi/2$. In GR we have $\eta = 1$ and hence $\Phi_{\text{eff}} = \Psi$. We caution that the definition (5.11) is valid only for $\Psi \neq 0$. If the gravitational potential Ψ crosses 0 with oscillations, we should compute Φ_{eff} from Eq. (5.10) rather than using η . As we will see in Sec. VI, the crossing of $\Psi = 0$ can actually occur in GLPV theories if the oscillating mode initially dominates the perturbation v_m .

B. The quasi-static approximation on sub-horizon scales

For the observations of large-scale structure and weak lensing, we are primarily interested in the evolution of perturbations for the modes deep inside the sound horizon ($c_s k \gg aH$). In the presence of a propagating scalar degree of freedom, there is an oscillating mode of the field perturbation in addition to the mode induced by matter perturbations. Provided that the oscillating mode of perturbations is suppressed relative to the matter-induced mode, the time derivatives of metric perturbations (like $\dot{\zeta}$ and $\dot{\psi}$) can be neglected relative to the terms involving their spatial derivatives [52]. In Horndeski theories, this quasi-static approximation was first employed in Ref. [19] to derive the analytic expression of G_{eff} , Φ_{eff} , and η .

In GLPV theories, let us discuss what kind of difference from Horndeski theories arises. First of all, Eq. (3.33) can be written in the form

$$(1 + \alpha_H)\Psi + c_t^2\zeta + \left(1 + \frac{\dot{L}_{,S}}{HL_{,S}}\right)H\psi = \alpha_H\dot{\psi}. \quad (5.12)$$

Since the time derivative $\alpha_H\dot{\psi}$ does not vanish, we need to deal with Eq. (5.12) as the differential equation rather than the constraint equation.

Under the quasi-static approximation on sub-horizon scales, the dominant contributions to Eq. (3.26) can be regarded as those involving the Laplacian terms $\partial^2\psi/a^2$, $\partial^2\zeta/a^2$ and $\delta\rho$. Then, in Fourier space, Eq. (3.26) reduces to

$$\mathcal{W}\frac{k^2}{a^2}\psi + 4L_{,S}(1 + \alpha_H)\frac{k^2}{a^2}\zeta \simeq \rho\delta. \quad (5.13)$$

Taking the time derivative of Eq. (3.26) and eliminating the term $\dot{\delta\rho}$ on account of Eq. (3.32), the quasi-static approximation for sub-horizon perturbations leads to

$$\mathcal{W}\Psi + (\dot{\mathcal{W}} + H\mathcal{W} + \rho)\psi + 4\left[(1 + \alpha_H)(\dot{L}_{,S} + HL_{,S}) + L_{,S}\dot{\alpha}_H\right]\zeta \simeq -4\alpha_H L_{,S}\dot{\zeta}. \quad (5.14)$$

In deriving Eq. (5.14) we have implicitly assumed that the mass m_ϕ of the scalar degree of freedom ϕ is at most of the order of the Hubble parameter H . In some of the modified gravity models in which the chameleon mechanism is at work [10], m_ϕ can be much larger than H as we go back to the past. In the regime $m_\phi \gg H$, however, the scalar degree of freedom is nearly frozen, so that the evolution of perturbations is similar to that in GR [19]. The modification of gravity manifests itself in the late cosmological epoch associated with the cosmic acceleration, in which regime m_ϕ is at most of the order of H .

In Horndeski theories we have $\alpha_H = 0$, so the terms on the r.h.s. of Eqs. (5.12) and (5.14) identically vanish. In this case, we can express ζ and ψ in terms of Ψ by using Eqs. (5.12) and (5.14). Substituting these relations into Eq. (5.13), we obtain the modified Poisson equation (5.7) with the effective gravitational coupling G_{eff} . The effective gravitational potential Φ_{eff} and the anisotropic parameter η are known accordingly. This procedure is given in Appendix A.

In GLPV theories the time derivatives $\dot{\psi}$ and $\dot{\zeta}$ are present, so we cannot derive the closed-form expression of G_{eff} and Φ_{eff} . The existence of the time-derivative terms in Eqs. (5.12) and (5.14) implies that the oscillating mode may play a non-trivial role for the evolution of perturbations. In Sec. VI we shall discuss the condition under which the oscillating mode is suppressed relative to the matter-induced mode for a model in the framework of GLPV theories.

VI. A CONCRETE MODEL

In this section we consider a concrete model in which Horndeski theories are minimally extended to GLPV theories. The model is described by the action (3.1) with the Lagrangian

$$L = A_2 + A_4(K^2 - S) + B_4\mathcal{R}, \quad (6.1)$$

where

$$A_2 = -\frac{1}{2}X - V(\phi), \quad A_4 = -\frac{1}{2}M_{\text{pl}}^2, \quad B_4 = \frac{1}{2}M_{\text{pl}}^2F(\phi), \quad (6.2)$$

with $V(\phi)$ and $F(\phi)$ being functions of ϕ . The model with $F(\phi) = 1$ correspond to GR in the presence of a minimally coupled scalar field ϕ with a potential $V(\phi)$. If the function $F(\phi)$ differs from 1, then the first condition of Eq. (2.5)

is not satisfied, so this is already beyond the realm of Horndeski theories. For the matter Lagrangian L_m we consider a single perfect fluid described by the constant equation of state $w = P/\rho$.

From Eqs. (3.14)-(3.15) we obtain the background equations of motion

$$3M_{\text{pl}}^2 H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho, \quad (6.3)$$

$$-2M_{\text{pl}}^2 \dot{H} = \dot{\phi}^2 + \rho + P. \quad (6.4)$$

Taking the time derivative of Eq. (6.3) and using Eqs. (3.16) and (6.4), the scalar field ϕ obeys

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \quad (6.5)$$

Equations (6.3)-(6.5) are equivalent to those in GR. This means that, at the background level, we cannot distinguish between the theories with same A_4 but with different B_4 [41].

For the later convenience we introduce the following variables [53]

$$x_1 \equiv \frac{\dot{\phi}}{\sqrt{6}HM_{\text{pl}}}, \quad x_2 \equiv \frac{\sqrt{V}}{\sqrt{3}HM_{\text{pl}}}, \quad \Omega_m \equiv \frac{\rho}{3M_{\text{pl}}^2 H^2}, \quad \lambda \equiv -\frac{M_{\text{pl}}V_{,\phi}}{V}, \quad (6.6)$$

with which the Friedmann equation (6.3) can be written as $\Omega_m = 1 - x_1^2 - x_2^2$. The dimensionless quantities x_1 , x_2 , and $x_3 \equiv \phi/M_{\text{pl}}$ obey the equations of motion

$$x_1' = -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 + \frac{3}{2}x_1 [(1-w)x_1^2 + (1+w)(1-x_2^2)], \quad (6.7)$$

$$x_2' = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{3}{2}x_2 [(1-w)x_1^2 + (1+w)(1-x_2^2)], \quad (6.8)$$

$$x_3' = \sqrt{6}x_1, \quad (6.9)$$

where a prime represents a derivative with respect to $\mathcal{N} = \ln a$. Note that the variable λ is known as a function of x_3 .

A. Propagation speeds

The background degeneracy between the theories with different values of B_4 is broken at the level of cosmological perturbations. From Eq. (3.35) the tensor propagation speed squared is given by

$$c_t^2 = -\frac{B_4}{A_4} = F(\phi). \quad (6.10)$$

The deviation from Horndeski theories can be quantified by the difference of c_t^2 from 1. In fact, the deviation parameter α_H in Eq. (4.8) reads

$$\alpha_H = c_t^2 - 1. \quad (6.11)$$

The quantities c_H^2 and β_H defined in Eqs. (4.7) and (4.9) reduce, respectively, to

$$c_H^2 = 1 + \frac{2M_{\text{pl}}^2 H^2}{\dot{\phi}^2} \left[(1 - c_t^2) \frac{\dot{H}}{H^2} + \frac{2c_t \dot{c}_t}{H} \right], \quad (6.12)$$

$$\beta_H = 2(1 - c_t^2) \left(1 + \frac{2M_{\text{pl}}^2 \dot{H}}{\dot{\phi}^2} \right), \quad (6.13)$$

where we used the background Eq. (6.4). In the regime $|\alpha_H| \ll 1$, the sound speed squared (4.13) can be estimated as $c_s^2 \simeq c_H^2 - \beta_H$, i.e.,

$$c_s^2 \simeq 1 - 2(1 - c_t^2) - \frac{2M_{\text{pl}}^2 H^2}{\dot{\phi}^2} \left[(1 - c_t^2) \frac{\dot{H}}{H^2} - \frac{2c_t \dot{c}_t}{H} \right]. \quad (6.14)$$

The density parameters of the field kinetic energy and the potential energy are given, respectively, by $\Omega_X = x_1^2$ and $\Omega_V = x_2^2$. In terms of these parameters, Eq. (6.4) can be expressed as

$$\frac{\dot{H}}{H^2} = -3\Omega_X - \frac{3}{2}(1+w)\Omega_m. \quad (6.15)$$

Then, the sound speed squared (6.14) reduces to

$$c_s^2 \simeq 1 - (c_t^2 - 1) \left[(1+w) \frac{\Omega_m}{2\Omega_X} - 1 \right] + \frac{2c_t \dot{c}_t}{3H\Omega_X}. \quad (6.16)$$

If the field ϕ is responsible for dark energy, its kinetic energy is usually suppressed relative to the background energy density in the early cosmological epoch, i.e., $\Omega_X/\Omega_m \ll 1$ (unless we consider early dark energy models). For $c_t^2 > 1$ the sound speed squared (6.16) can be negative, which leads to the Laplacian instability on small scales. Let us consider the case in which $\dot{c}_t = 0$, i.e., $F(\phi) = \text{constant}$. In order to realize the condition $c_s^2 > 0$ during the radiation and early matter eras, we require that

$$c_t^2 - 1 \lesssim \frac{\Omega_X}{\Omega_m}. \quad (6.17)$$

As we go back to the past the ratio Ω_X/Ω_m gets smaller, so c_t^2 needs to be very close to 1 in the early cosmological era.

When $c_t^2 < 1$ the Laplacian instability is absent, but c_s becomes highly super-luminal. This means that the quantity $c_s k/a$, which appears in the perturbation equations of motion, becomes much larger than k/a , so that the perturbation theory can break down for $c_s k/a$ above some cut-off scale M .

We stress that the problem of large values of $|c_s^2|$ persists whenever there is a scalar field ϕ whose kinetic energy is suppressed relative to the background energy density. The slowly varying scalar potentials $V(\phi)$ responsible for the late-time cosmic acceleration are generally plagued by this problem.

One way out is to consider a primordial scaling field characterized by the constant ratio Ω_X/Ω_m [53–55]. The scaling solution can be realized by the exponential potential $V(\phi) = V_1 e^{-\lambda_1 \phi/M_{\text{Pl}}}$ for the constant λ_1 satisfying $\lambda_1^2 > 3(1+w)$ [53]. For the success of the big bang nucleosynthesis, the slope λ_1 is constrained to be $\lambda_1 > 9.4$ [56]. In this case the late-time cosmic acceleration is not realized, so the form of the potential needs to be modified to exit from the scaling regime. One of such models is given by [57]

$$V(\phi) = V_1 e^{-\lambda_1 \phi/M_{\text{Pl}}} + V_2 e^{-\lambda_2 \phi/M_{\text{Pl}}}, \quad (6.18)$$

where $V_1, V_2, \lambda_1, \lambda_2$ are constants. Provided that $\lambda_2^2 < 2$, the solutions finally approach the acceleration attractor characterized by the field equation of state $w_{\text{DE}} = -1 + \lambda_2^2/3$.

B. Perturbation equations and the oscillating mode

Since our primary interest is the growth of structures after the onset of the matter-dominated epoch, we consider a non-relativistic perfect fluid characterized by $w = 0$ and $c_m^2 = 0$ for the matter Lagrangian $L_m = P(Y)$. For example, the Lagrangian $P(Y) = c_2(Y - Y_0)^2$ with $|(Y - Y_0)/Y_0| \ll 1$ can describe a non-relativistic perfect fluid [41, 46]. In the following we also focus on the case in which c_t^2 is constant.

First, we introduce the following dimensionless quantities

$$V_m \equiv H v_m, \quad \chi \equiv H \psi. \quad (6.19)$$

For the model (6.1), the perturbation equations of motion in Fourier space following from Eqs. (3.26), (3.27), and

(3.31)-(3.33) are given by

$$\zeta' = \delta N + \frac{3}{2}\Omega_m V_m, \quad (6.20)$$

$$\chi' = \left(\frac{H'}{H} - 1\right)\chi - c_t^2 \delta N - c_t^2 \zeta, \quad (6.21)$$

$$\delta' = -\frac{3}{2}\Omega_m \delta + 3(\Omega_X - 1)\delta N + \left(\frac{k}{aH}\right)^2 (c_t^2 \zeta + V_m), \quad (6.22)$$

$$V_m' = -\delta N + \frac{H'}{H} V_m, \quad (6.23)$$

$$\Omega_X \delta N = \frac{1}{2}\Omega_m \delta_m - \frac{1}{3}\left(\frac{k}{aH}\right)^2 (c_t^2 \zeta + \chi), \quad (6.24)$$

where $\delta_m = \delta - 3V_m$ is the gauge-invariant matter perturbation.

The two gravitational potentials Ψ and Φ are given, respectively, by $\Psi = \delta N - (H'/H)\chi + \chi'$ and $\Phi = \zeta + \chi$. From Eq. (6.21) we obtain the following relation

$$\Psi + \Phi = (1 - c_t^2)(\delta N + \zeta). \quad (6.25)$$

Taking the \mathcal{N} derivative of Eqs. (6.22)-(6.23) and using other equations of motion, it follows that

$$\delta_m'' + \left(2 + \frac{H'}{H}\right)\delta_m' - \frac{3}{2}\Omega_m \delta_m = \left(\frac{k}{aH}\right)^2 (c_t^2 - 1)\delta N - 12\Omega_X \delta N + 18\Omega_X \left(1 + \frac{H'}{H} + \frac{\Omega_X'}{2\Omega_X}\right)V_m. \quad (6.26)$$

Since $\Psi = -\Phi$ for $c_t^2 = 1$, the anisotropic parameter η defined by Eq. (5.11) is equivalent to 1. In this case the first term on the r.h.s. of Eq. (6.26) also vanishes. Provided $\Omega_X \ll 1$, the remaining terms on the r.h.s. of Eq. (6.26) are suppressed relative to those on the l.h.s., so we obtain the growing-mode solution $\delta_m \propto a$ in the deep matter era ($H'/H \simeq -3/2$ and $\Omega_m \simeq 1$).

In GLPV theories, there is the anisotropic stress ($\eta \neq 1$) between the gravitational potentials induced by the difference of c_t^2 from 1. Unlike the usual modified gravity models, the parameter η deviates from 1 even in the early cosmological epoch. We also note that the existence of the first term on the r.h.s. of Eq. (6.26) leads to the modified growth of matter perturbations relative to the case $c_t^2 = 1$.

In order to understand the evolution of the velocity potential, we take the \mathcal{N} derivative of Eq. (6.23) and then use other equations of motion. The resulting second-order equation for V_m reads

$$V_m'' + \alpha_1 V_m' + \alpha_2 V_m = -\left(\frac{k}{aH}\right)^2 \chi, \quad (6.27)$$

where

$$\alpha_1 \equiv \frac{\sqrt{6}}{4x_1} \left[4\lambda(1 - \Omega_m - x_1^2) - \sqrt{6}x_1(2 - \Omega_m - 2x_1^2) \right], \quad (6.28)$$

$$\begin{aligned} \alpha_2 \equiv & -(c_t^2 - 1)\frac{\Omega_m}{2x_1^2} \left(\frac{k}{aH}\right)^2 \\ & + \frac{3}{2x_1} \left[3x_1\{4x_1^4 + 2(2\Omega_m - 3)x_1^2 + \Omega_m(\Omega_m - 1)\} - \sqrt{6}\lambda(\Omega_m + 4x_1^2)(\Omega_m + x_1^2 - 1) \right]. \end{aligned} \quad (6.29)$$

The general solution to Eq. (6.27) can be expressed in the following form

$$V_m = V_m^{(s)} + V_m^{(h)}, \quad (6.30)$$

where $V_m^{(s)}$ is the special solution and $V_m^{(h)}$ is the homogeneous solution derived by setting the r.h.s. of Eq. (6.27) to be 0. As long as the first term on the r.h.s. of Eq. (6.29) dominates over the other terms, the special solution is given by

$$V_m^{(s)} \simeq \frac{1}{c_t^2 - 1} \frac{2\Omega_X}{\Omega_m} \chi = -\frac{1}{c_s^2 - c_t^2} \chi. \quad (6.31)$$

In the second equality we used the fact that, for $c_m^2 = 0$, the sound speed squared c_s^2 is exactly given by Eq. (6.16) with $w = 0$.

The oscillations of perturbations are induced by the homogenous solution $V_m^{(h)}$. In order to see the behavior of oscillations, we consider the scaling solution during the matter-dominated epoch. The scaling matter era can be realized by the dominance of the potential $V_1 e^{-\lambda_1 \phi/M_{\text{Pl}}}$ in Eq. (6.18), which corresponds to [53]

$$x_1 = x_2 = \frac{\sqrt{6}}{2\lambda_1}, \quad \Omega_m = 1 - \frac{3}{\lambda_1^2}. \quad (6.32)$$

Substituting Eq. (6.32) into Eqs. (6.28)-(6.29) with $\lambda = \lambda_1$, we find that the homogenous solution satisfies

$$V_m^{(h)''} + \frac{3}{2}V_m^{(h)'} + \frac{1}{3}(\lambda_1^2 - 3) \left[(1 - c_t^2) \left(\frac{k}{aH} \right)^2 + \frac{27}{2\lambda_1^2} \right] V_m^{(h)} = 0. \quad (6.33)$$

During the scaling matter era the scale factor evolves as $a \propto t^{2/3}$, so the evolution of the quantity

$$K \equiv \frac{k}{aH} \quad (6.34)$$

is known as $K(\mathcal{N}) = K_i e^{\mathcal{N}/2}$, where K_i is initial value of K at $\mathcal{N} = 0$. Then Eq. (6.33) can be expressed as

$$U_m^{(h)''} + \left[c_{\text{eff}}^2 K_i^2 e^{\mathcal{N}} + \frac{9(7\lambda_1^2 - 24)}{16\lambda_1^2} \right] U_m^{(h)} = 0, \quad (6.35)$$

where

$$U_m^{(h)} \equiv a^{3/4} V_m^{(h)}, \quad c_{\text{eff}}^2 \equiv \frac{1}{3}(\lambda_1^2 - 3)(1 - c_t^2). \quad (6.36)$$

If $c_t^2 > 1$, then we have $c_{\text{eff}}^2 < 0$ and hence the perturbation $U_m^{(h)}$ is prone to the Laplacian instability on small scales. When $c_t^2 < 1$, the perturbation exhibits oscillations induced by the positive Laplacian term $c_{\text{eff}}^2 K^2(\mathcal{N})$. Note that c_{eff}^2 is related to c_s^2 as $c_{\text{eff}}^2 = c_s^2 - c_t^2$. As long as $c_s^2 \gg c_t^2$, c_{eff}^2 is approximately equivalent to c_s^2 .

The solution to Eq. (6.35) in the regime $c_t^2 < 1$ is given by

$$U_m^{(h)} = c_1 J_\nu(x) + c_2 Y_\nu(x), \quad (6.37)$$

where c_1, c_2 are integration constants, $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of first and second kinds respectively, with

$$\nu \equiv \frac{3\sqrt{7\lambda_1^2 - 24}}{2\lambda_1} i, \quad x \equiv 2c_{\text{eff}} K(\mathcal{N}) = \frac{2c_{\text{eff}} k}{aH}. \quad (6.38)$$

Provided that $x \gg 1$, the first term in the square bracket of Eq. (6.35) is much larger than the second term (which is of the order of 1). In this case the solution to Eq. (6.35) is given by Eq. (6.37) with the index $\nu \simeq 0$. Using the asymptotic forms of the Bessel functions in the limit $x \gg 1$, we obtain the following approximate solution

$$V_m^{(h)} \simeq a^{-3/4} \sqrt{\frac{2}{\pi x}} \left[c_1 \cos\left(x - \frac{\pi}{4}\right) + c_2 \sin\left(x - \frac{\pi}{4}\right) \right]. \quad (6.39)$$

The perturbation $V_m^{(h)}$ exhibits a damped oscillation with the frequency determined by $c_{\text{eff}} k$. Since $x \propto e^{\mathcal{N}/2}$ in the scaling matter era, the amplitude decreases as $|V_m^{(h)}| \propto a^{-1}$. As we will see later, the homogenous solution (6.31) stays nearly constant during the matter era. This means that, as we go back to the past, the oscillating solution (6.39) tends to dominate over the homogenous solution (as it happens for the dark energy models in $f(R)$ theories [10]). The dominance of the oscillating solution can be avoided for the initial conditions satisfying $|V_m^{(h)}| \ll |V_m^{(s)}|$. This amounts to choosing the initial conditions of V_m close to the value (6.31) and $V_m' \simeq 0$.

C. Evolution of perturbations

Numerically we integrate the perturbation equations of motion (6.20)-(6.23) with (6.24) in order to find the precise evolution of V_m , δ_m as well as the gravitational potentials Ψ and Φ .

First, we study the case in which the special solution to Eq. (6.27) dominates over the homogenous solution (6.39), notwithstanding that this requires fine-tuning of the initial conditions for the perturbations. The initial conditions corresponding to such a case are given by $V_m \simeq -K^2\chi/\alpha_2$ and $V'_m \simeq 0$. From Eq. (6.23) the latter condition implies that $\delta N \simeq (H'/H)V_m$, in which case the r.h.s. of Eq. (6.20) is small such that $\zeta' \simeq 0$. As in the case of GR, we also employ the initial condition $\chi' = 0$. Then, for given δ_m , the initial conditions of ζ , χ , V_m , and δ are known accordingly.

In Fig. 1 we plot the evolution of V_m , $-\Psi$, and Φ versus the redshift $z = a_0/a - 1$ (a_0 is the today's value of a) for $c_t^2 = 0.5$, $\lambda_1 = 10$, $\lambda_2 = 0$, and $V_2/V_1 = 10^{-6}$. During the matter-dominated epoch, the background solutions are in the scaling regime characterized by Eq. (6.32). Numerically we find that the perturbations V_m , ζ , χ , and δN stay nearly constant by the end of the matter era for the initial conditions explained above. On using the approximate relations (6.31) as well as $V'_m \simeq 0$ and $\chi' \simeq 0$ with $H'/H = -3/2$, the evolution of perturbations during the scaling matter era can be estimated as

$$\begin{aligned} V_m &\simeq \frac{\Omega_m \delta_m}{c_s^2 K^2 - 3\Omega_X}, & \delta N &\simeq -\frac{3}{2} \frac{\Omega_m \delta_m}{c_s^2 K^2 - 3\Omega_X}, \\ \zeta &\simeq \frac{5c_s^2 - 2c_t^2}{2c_t^2(c_s^2 K^2 - 3\Omega_X)} \Omega_m \delta_m, & \chi &\simeq -\frac{c_s^2 - c_t^2}{c_s^2 K^2 - 3\Omega_X} \Omega_m \delta_m. \end{aligned} \quad (6.40)$$

Then the gauge-invariant gravitational potentials satisfy

$$\Psi \simeq -\frac{3(1 + c_s^2 - c_t^2)}{2(c_s^2 K^2 - 3\Omega_X)} \Omega_m \delta_m, \quad \Phi \simeq \frac{5c_s^2 - 2c_t^2(1 + c_s^2 - c_t^2)}{2c_t^2(c_s^2 K^2 - 3\Omega_X)} \Omega_m \delta_m, \quad (6.41)$$

so that the anisotropic parameter is simply given by

$$\eta \simeq 1 + \frac{5(1 - c_t^2)(c_s^2 - c_t^2)}{3c_t^2(1 + c_s^2 - c_t^2)}, \quad (6.42)$$

where $c_s^2 = c_t^2 + (\lambda_1^2 - 3)(1 - c_t^2)/3$.

We confirmed that the analytic solutions (6.40)-(6.41) show good agreement with the numerical results shown in Fig. 1. In Fig. 2 we plot the evolution of η for three different values of c_t^2 smaller than 1. The anisotropic parameter is nearly constant during the scaling matter era. As estimated by Eq. (6.42), the deviation of c_t^2 from 1 leads to the values of η larger than 1. This is the observational signature of our model manifest in CMB temperature anisotropies [58]. From Fig. 1 we find that the two gravitational potentials $-\Psi$ and Φ start to decrease after the onset of the cosmic acceleration ($z \lesssim 1$). As we see in Fig. 2, this leads to the variation of η , which signals the end of the scaling matter era.

Substituting the solutions (6.40) into Eq. (6.26), the perturbation δ_m during the scaling matter era obeys the following equation

$$\delta_m'' + \frac{1}{2}\delta_m' - \frac{3}{2}\frac{G_{\text{eff}}}{G}\Omega_m\delta_m \simeq 0, \quad G_{\text{eff}} = \frac{K^2(c_s^2 + 1 - c_t^2) + 3\Omega_X}{c_s^2 K^2 - 3\Omega_X} G. \quad (6.43)$$

As long as $c_s^2 K^2 \gg \Omega_X$, the effective gravitational coupling reduces to

$$G_{\text{eff}} \simeq \left(1 + \frac{1 - c_t^2}{c_s^2}\right) G. \quad (6.44)$$

When $c_t^2 < 1$ we have $G_{\text{eff}} > G$. On using Eq. (6.44), the growing-mode solution to Eq. (6.43) is given by

$$\delta_m \propto a^p, \quad p = \frac{1}{4} \sqrt{1 + 24\Omega_m + 24\Omega_m \frac{1 - c_t^2}{c_s^2}} - \frac{1}{4}. \quad (6.45)$$

Compared to the case of GR ($c_t^2 = 1$), the growth rate of matter perturbations gets larger for $c_t^2 < 1$. However, since $c_s^2 = c_t^2 + (\lambda_1^2 - 3)(1 - c_t^2)/3$ and $\lambda_1 \gtrsim 10$, this modification is suppressed to be small. For example, even for $c_t^2 = 0$, we have $24\Omega_m(1 - c_t^2)/c_s^2 = 72/\lambda_1^2 \lesssim 0.7$ in Eq. (6.45), so the growth index p is not very different from that for $c_t^2 = 1$.

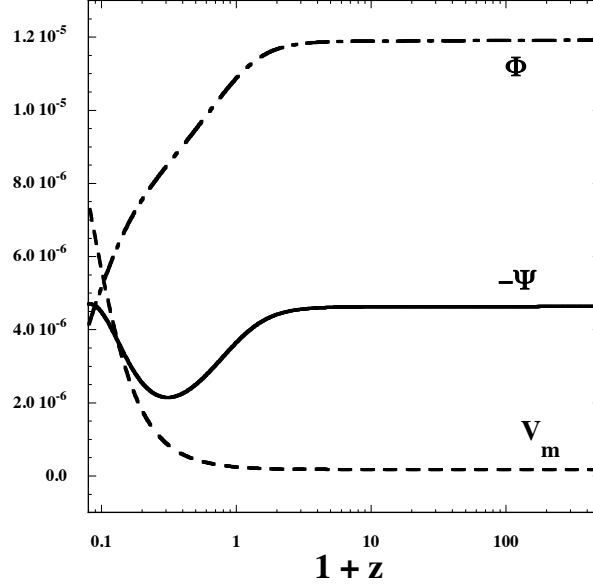


Figure 1: The evolution of the velocity potential V_m and the gravitational potentials $-\Psi$ and Φ versus $1+z$ ($= a_0/a$) for $c_t^2 = 0.5$, $\lambda_1 = 10$, $\lambda_2 = 0$, and $V_2/V_1 = 10^{-6}$. The present epoch ($z = 0$) is identified as $\Omega_m = 0.3$. We choose the initial conditions as $x_1 = x_2 = \sqrt{6}/(2\lambda_1)$, $x_3 = 0$, $\zeta = 1.4832 \times 10^{-5}$, $\chi = -2.9124 \times 10^{-6}$, $\delta = 1.9351 \times 10^{-3}$, and $V_m = 1.8007 \times 10^{-7}$ for the wave number $K = k/(aH) = 25$ at the redshift $z = 527.9$. These correspond to the initial conditions with the negligible oscillating mode characterized by $V_m = -K^2 \chi / \alpha_2$ and $V'_m = 0$.

In Fig. 3 we show the evolution of $f\sigma_8$ for three different values of c_t^2 . When $c_t^2 < 1$ the effective gravitational coupling (6.44) is bigger than G , so the growth rate of δ_m tends to be slightly larger than that for $c_t^2 = 1$. However, the difference between the two cases $c_t^2 = 0.1$ and $c_t^2 = 1$ is very tiny except for the low-redshift regime in which the cosmic acceleration occurs.

In the super-luminal regime $c_t^2 > 1$, the Laplacian instability associated with negative c_s^2 can be avoided for

$$c_t^2 - 1 < \frac{2\Omega_X}{\Omega_m}. \quad (6.46)$$

During the scaling matter era, this condition translates to $c_t^2 - 1 < 3/(\lambda_1^2 - 3)$. The dashed curve shown in Fig. 3 ($c_t^2 = 1.02$) is the case in which the stability condition (6.46) is satisfied. In the low-redshift regime, $f\sigma_8$ for $c_t^2 = 1.02$ is slightly smaller than that for $c_t^2 = 1$, so the gravitational force tends to be weaker. Numerically we confirmed that, for c_t^2 violating the condition (6.46), the perturbations are prone to violent negative instabilities.

Finally, we discuss the case in which the initial condition $|V_m^{(h)}| \ll |V_m^{(s)}|$ is violated. The numerical simulation shown in Fig. 4 corresponds to such an example, where the initial value of V_m is not close to the special solution (6.31). When $c_t^2 < 1$, the velocity perturbation V_m exhibits an oscillation induced by the sound speed c_s during the scaling matter era. The amplitude of the homogenous solution decreases as $|V_m^{(h)}| \propto a^{-1}$, so V_m approaches the special solution (6.31) in the end. Provided that the approach to $V_m^{(s)}$ occurs for $z \gg 1$, the evolution of $f\sigma_8$ in the low-redshift regime is similar to that shown in Fig. 3.

In the numerical simulation of Fig. 4, we find that the gravitational potential $-\Psi$ shows a heavy oscillation around 0 with a large amplitude. We recall that Ψ is related to the perturbation $\delta N = -V'_m + (H'/H)V_m$, as $\Psi = \delta N - (H'/H)\chi + \chi'$. From Eq. (6.39) the derivative $V_m^{(h) '}$ can be estimated as

$$V_m^{(h) ' \simeq a^{-3/4} \sqrt{\frac{x}{2\pi}} \left[-c_1 \sin \left(x - \frac{\pi}{4} \right) + c_2 \cos \left(x - \frac{\pi}{4} \right) \right], \quad (6.47)$$

where we have assumed $x = 2c_{\text{eff}}K(\mathcal{N}) \gg 1$. The amplitude of $V_m^{(h) '}$ is $x/2$ times as large as that of $V_m^{(h)}$. This is the reason why the amplitude of the oscillating mode of $-\Psi$ is much larger than that of V_m . The definition of the anisotropic parameter $\eta = -\Phi/\Psi$ loses its validity whenever $-\Psi$ crosses 0, so in such cases we should directly resort to the effective gravitational potential (5.10) rather than η . Furthermore, as $-\Psi$ crosses 0, the effective gravitational

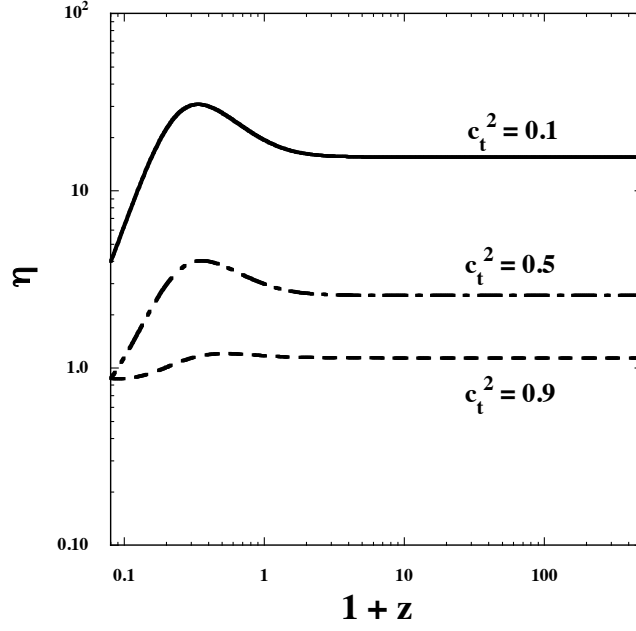


Figure 2: The evolution of the anisotropic parameter η versus $1+z$ for $c_t^2 = 0.1, 0.5, 0.9$ with the model parameters $\lambda_1 = 10$, $\lambda_2 = 0$, and $V_2/V_1 = 10^{-6}$. The initial conditions are chosen to be similar to those given in Fig. 1, such that $|V_m^{(h)}| \ll |V_m^{(s)}|$.

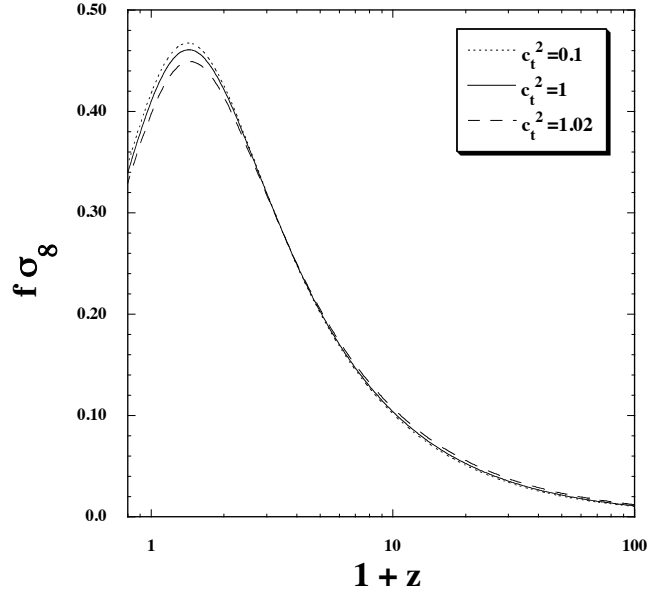


Figure 3: The evolution of $f\sigma_8$ versus $1+z$ for $c_t^2 = 0.1, 1, 1.02$ with the model parameters $\lambda_1 = 10$, $\lambda_2 = 0$, and $V_2/V_1 = 10^{-6}$. The initial conditions are chosen to be similar to those given in Fig. 1, such that $|V_m^{(h)}| \ll |V_m^{(s)}|$.

coupling G_{eff} derived from Eq. (5.7) changes its sign, leading, in this case, to a rather unclear interpretation for the gravitational interaction. In the numerical simulation of Fig. 4, the oscillation of $-\Psi$ does not damp away even around today ($z \sim 1$). The amplitude of the oscillating mode of $\Phi = \zeta + \chi$ is smaller than that of $-\Psi$, such that the oscillation of the former disappears in the early stage of the matter era (see Fig. 4).

The oscillation of the gravitational potential $-\Psi$ leaves a distinctive imprint in CMB temperature anisotropies, so it should be possible to put tight constraints on the parameter space of initial conditions. We leave the analysis of observational constraints on the model (including the case of initial conditions $|V_m^{(h)}| \ll |V_m^{(s)}|$) for a future work.

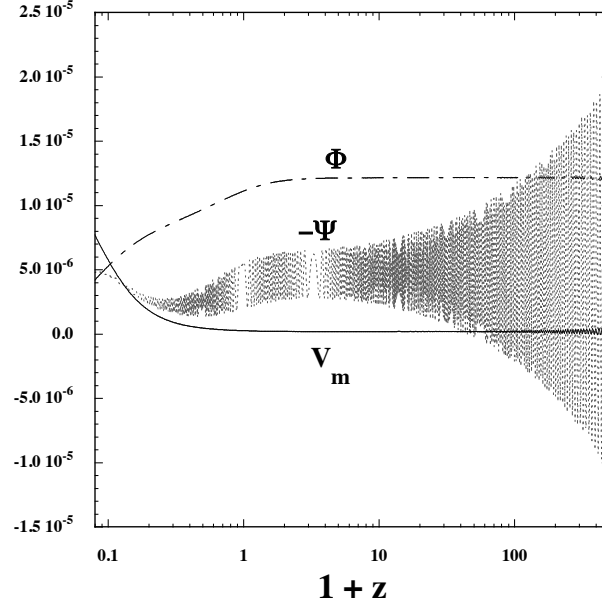


Figure 4: The evolution of the velocity potential V_m and the gravitational potentials $-\Psi$ and Φ versus $1+z$ for $c_t^2 = 0.5$, $\lambda_1 = 10$, $\lambda_2 = 0$, and $V_2/V_1 = 10^{-6}$ with the initial value $V_m = 5.0 \times 10^{-7}$ and other initial conditions same as those in Fig. 1.

VII. CONCLUSIONS

In the framework of the EFT of modified gravity, we have studied the dynamics of cosmological perturbations and resulting observational consequences. Our starting point is the general action (3.1) in unitary gauge that involves a gravitational scalar degree of freedom and a matter field χ mimicking a perfect fluid. This analysis encompasses the GLPV theories described by the Lagrangian (2.4) as a specific case.

Expanding the action up to second order in the perturbations of ADM scalar quantities, we derived the scalar perturbation equations of motion (3.26)-(3.29) as well as the equation (3.36) for gravitational waves. In GLPV theories there is a non-trivial mixing between the propagation speeds of matter and gravitational scalar. The matter sound speed is not strongly affected by the deviation from Horndeski theories, but the modification to the scalar sound speed c_s is large enough to be able to distinguish between GLPV and Horndeski theories. For non-relativistic matter with $c_m^2 = 0$ we derived the exact relation $c_s^2 = c_H^2 - \beta_H^2$, where c_H is given by Eq. (4.7) and β_H is a new term (4.9) arising in the theories beyond Horndeski.

As an application of our general formalism, we studied the cosmology for a simple canonical scalar-field model described by the Lagrangian (6.1). The deviation from Horndeski theories can be quantified by the difference of the tensor propagation speed squared $c_t^2 = F(\phi)$ from 1, where $F(\phi)$ is a function of ϕ appearing in Eq. (6.2). In the regime $|\alpha_H| = |c_t^2 - 1| \ll 1$, the scalar sound speed squared c_s^2 is given by Eq. (6.16). For dark energy models in which the ratio Ω_m/Ω_X gets larger in the past (which is the case for most of models), $|c_s^2|$ can be much larger than 1 even for $|\alpha_H| \ll 1$. In particular, for $c_t^2 > 1$, the scalar perturbation is plagued by the Laplacian instability unless the condition (6.17) is satisfied.

The behavior of large values of c_s^2 in the past can be avoided for the dark energy model given by the potential (6.18), in which case there exists the scaling solution characterized by the constant ratio $\Omega_m/\Omega_X = 2(\lambda_1^2 - 3)/3$. In this model we studied the evolution of cosmological perturbations to find observational signatures in the framework of GLPV theories. The solution to Eq. (6.27) for the velocity potential $V_m = H v_m$ can be written in terms of the sum of the special solution $V_m^{(s)}$ and the homogenous solution $V_m^{(h)}$. During the scaling matter era $V_m^{(s)}$ stays nearly constant, whereas $V_m^{(h)}$ decreases in proportion to a^{-1} with oscillations.

Provided that $|V_m^{(h)}| \ll |V_m^{(s)}|$, the evolution of perturbations during the scaling matter era can be known analytically. Of course this requires the fine-tuning of initial conditions. In this case, the anisotropic parameter $\eta = -\Phi/\Psi$ between the two gravitational potentials is expressed in terms of c_t^2 and c_s^2 alone. As we see in Fig. 2, the deviation of η from 1 tends to be larger as c_t^2 is away from 1. We have also estimated the growth rate of matter perturbations δ_m and found that the evolution of δ_m is not sensitive to the tensor propagation speed ranging in the regime (6.46).

If $V_m^{(h)}$ dominates over $V_m^{(s)}$ at the initial stage of the matter era, the perturbations exhibit rapid oscillations with

the frequency related to c_s^2 until V_m approaches the special solution $V_m^{(s)}$. The amplitude of oscillations is particularly large for the gravitational potential Ψ , see Fig. 4. This property comes from the fact that Ψ is related to the derivative term $V_m^{(h)'}$, whose amplitude is $x/2$ times as large as that of $V_m^{(h)}$.

We have thus shown that the simple extension of Horndeski theories to GLPV theories gives rise to interesting observational signatures. For the initial conditions satisfying $|V_m^{(h)}| \ll |V_m^{(s)}|$, the anisotropic parameter η deviates from 1 even in the early stage of the matter era (which is usually not the case in Horndeski theories). For other initial conditions, the gravitational potential Ψ rapidly oscillates with a large amplitude especially when c_t^2 is away from 1. In both cases, we should be able to put tight bounds on the deviation parameter $\alpha_H = c_t^2 - 1$ from the observations of CMB and weak lensing. It will also of interest to study local gravity constraints on our model along the lines of Refs. [59–61].

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Appendix A: Quasi-static approximation on sub-horizon scales in Horndeski theories

In this Appendix we derive the expression of G_{eff} and Φ_{eff} in Horndeski theories under the quasi-static approximation for the modes deep inside the sound horizon. Since $\alpha_H = 0$ in this case, we can solve Eqs. (5.12) and (5.14) for ψ and ζ , as

$$\psi = \frac{L_{,S}(4b_1 - c_t^2\mathcal{W})}{b_2c_t^2L_{,S} - 4b_1^2}\Psi, \quad \zeta = \frac{b_1\mathcal{W} - b_2L_{,S}}{b_2c_t^2L_{,S} - 4b_1^2}\Psi, \quad (\text{A1})$$

where

$$b_1 \equiv \dot{L}_{,S} + HL_{,S}, \quad b_2 \equiv \dot{\mathcal{W}} + H\mathcal{W} + \rho. \quad (\text{A2})$$

Substituting the relations (A1) into Eq. (5.13) with the approximation $\delta_m \simeq \delta$, we obtain Eq. (5.7) with the effective gravitational coupling

$$G_{\text{eff}} = \frac{b_2c_t^2L_{,S} - 4b_1^2}{4\pi L_{,S}(\mathcal{W}^2c_t^2 + 4b_2L_{,S} - 8b_1\mathcal{W})}. \quad (\text{A3})$$

From Eq. (A1) the gravitational potential $\Phi = \zeta + H\psi$ satisfies the relation $\Phi = -\eta\Psi$, with the anisotropic parameter

$$\eta = \frac{b_2L_{,S} - b_1\mathcal{W} + HL_{,S}(\mathcal{W}c_t^2 - 4b_1)}{b_2c_t^2L_{,S} - 4b_1^2}. \quad (\text{A4})$$

On using Eq. (5.7), (A3), and (A4), the effective gravitational potential (5.10) obeys

$$\Phi_{\text{eff}} = -\frac{a^2}{k^2} \frac{b_2(1 + c_t^2)L_{,S} - 4b_1^2 - b_1\mathcal{W} + HL_{,S}(\mathcal{W}c_t^2 - 4b_1)}{2L_{,S}(\mathcal{W}^2c_t^2 + 4b_2L_{,S} - 8b_1\mathcal{W})} \rho \delta_m. \quad (\text{A5})$$

The results (A3)-(A5) match with those derived in Ref. [19] in the Newtonian gauge by taking the massless limit $m_\phi \rightarrow 0$.

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